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Modules of third-order differential operators on a conformally flat manifold

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Abstract

Let M be a smooth manifold endowed with a flat conformal structure and $\mathcal{F}_\lambda(M)$ the space of densities of degree λ on M. We study the space $\mathcal{D}^3_{\lambda,\mu}(M)$ of third-order differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$ as a module over the conformal Lie algebra $\mathrm{o}(p+1,q+1)$. We prove that $\mathcal{D}^3_{\lambda,\mu}(M)$ is isomorphic to the corresponding module of third-order polynomials on $T^*(M)$ for almost all values of $\delta=\mu-\lambda$, except for eight resonant values. The isomorphism is unique and will be given explicitly, yielding a conformally equivariant quantization. We also study the modules in the case of resonance.

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1. Introduction

1.1. Modules of differential operators

The study of the space $\mathcal{D}_{\lambda,\mu}$ of linear differential operators on a smooth manifold viewed as a module over the group $\mathrm{Diff}(M)$ of diffeomorphisms of M, and the Lie algebra $\mathrm{Vect}(M)$ of vectors fields on M is a classical problem of differential geometry [17]. The modules $\mathcal{D}_{\lambda,\mu}$ have already been considered in the classical literature on differential operators and, more recently, in a series of papers [3,5,10,12–14]. There is a filtration $\mathcal{D}^0_{\lambda,\mu}\subset\mathcal{D}^1_{\lambda,\mu}\subset\mathcal{D}^2_{\lambda,\mu}\subset\cdots\subset\mathcal{D}^k_{\lambda,\mu}\subset\cdots$, where $\mathcal{D}^0_{\lambda,\mu}\cong\mathcal{F}_{\mu-\lambda}$ is given by multiplication with $(\mu-\lambda)$-densities. The higher-order are defined by induction: $A\in\mathcal{D}^k_{\lambda,\mu}$ if $[A,f]\in\mathcal{D}^{k-1}_{\lambda,\mu}$ for every $f\in C^\infty(M)$.

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Let $S := \mathrm{Pol}(T^*M)$ be the space of polynomials on T^*M. This space is isomorphic to the space of contravariant symmetric tensor fields on M. One defines a one parameter family of $\mathrm{Diff}(M)$-modules or $\mathrm{Vect}(M)$-modules on the space of symbols

$$S_\delta := S \otimes \mathcal{F}_\delta.$$

This space is naturally considered (cf. [5,7]) as the space of symbols of $\mathcal{D}_{\lambda,\mu}$ for $\mu - \lambda = \delta$.

Again, there is a filtration $S_\delta^0 \subset S_\delta^1 \subset \cdots \subset S_\delta^k \subset \cdots$, where S_δ^k denotes the space of symbols of degree less or equal to k. In contrast to the filtration on $\mathcal{D}_{\lambda,\mu}$, the previous filtration on the space of symbols actually leads to a $\mathrm{Diff}(M)$-invariant graduation:

$$S_\delta = \bigoplus_{k=0}^{\infty} S_{k,\delta}.$$

Here $S_{k,\delta}$ denotes the space of homogenous polynomials (isomorphic to $S_\delta^k/S_\delta^{k-1}$). Let us recall that one result of [12] is that if $k \geq 3$ and $\dim M \geq 2$, the modules $\mathcal{D}_{\lambda,\mu}^k$ and $\mathcal{D}_{\lambda',\mu'}^k$ are isomorphic provided that $(\lambda', \mu') = (1-\mu, 1-\lambda)$. Moreover, $\mathcal{D}_{\lambda,\mu}^k$ is not isomorphic to the corresponding symbols space $S_\delta^k = \oplus_{l \leq k} S_{l,\delta}$ (cf. [13]).

1.2. Main problem

S_δ and $\mathcal{D}_{\lambda,\mu}$ *are neither isomorphic* as $\mathrm{Diff}(M)$-modules nor as $\mathrm{Vect}(M)$-modules. It is natural to consider a subgroup $G \subset \mathrm{Diff}(M)$ (of finite dimension) and restrict the action of $\mathrm{Diff}(M)$ to G. This idea has been used in [17] (for $\mathrm{SL}(2, \mathbf{R}) \subset \mathrm{Diff}(S^1)$) and [3,4,9] for the one dimensional case. In higher dimensions, the space $\mathcal{D}_{\lambda,\mu}^k$ has been studied in [6,7] on a conformally flat manifold. In this case the symmetry group is $G = \mathrm{SO}(p+1, q+1)$, $p+q = \dim(M)$. The main result of [7] is that $\mathcal{D}_{\lambda,\mu}^k$ and S_δ^k are isomorphic as $\mathrm{SO}(p+1, q+1)$-modules for almost all values of $\delta = \mu - \lambda$, except for resonant values. The detailed study of $\mathcal{D}_{\lambda,\mu}^2 \overset{\cong}{\rightarrow} S_\delta^2$ was performed in [6]. Here we propose the study of the isomorphism

$$\mathcal{D}_{\lambda,\mu}^3 \overset{\cong}{\rightarrow} S_\delta^3. \tag{1.1}$$

The conformally equivariant quantization was developed in [6] and the existence and uniqueness was proven in [7] for generic values of $\mu - \lambda$. Detailed studies of the modules of third-order linear differential operators are of particular interest; they yield interesting examples for quantization. In other word, the resonant modules whose existence was proven in [6,7] have not yet been studied in the case of third-order. However, these particular modules provide remarkable examples of differential operators.

1.3. Application to the quantization of the geodesic flow

As an example, the quantization of the geodesic flow yields a novel conformally equivariant Laplace operator on half-densities, as well as the well-known Yamabe Laplacian.

2. Main results

Let M be a manifold endowed with a flat conformal structure: there exists a local action of the group $\mathrm{O}(p+1, q+1)$ on M, which enables us to restrict the $\mathrm{Diff}(M)$-module $\mathcal{D}_{\lambda,\mu}$ to the conformal group. In the following, we recall the corresponding action of the Lie algebra $\mathrm{o}(p+1, q+1)$.

2.1. The conformal Lie algebra $\mathrm{o}(p+1, q+1) \subset \mathrm{Vect}(\mathbb{R}^n)$

It is well-known that a conformally flat manifold admits an atlas in which $\mathrm{o}(p+1, q+1)$ is generated by

$$X_i = \partial_i \quad \text{(translations)}, \qquad X_{ij} = x_i\partial_j - x_j\partial_i \quad \text{(rotations)},$$
$$\overline{X_i} = x_j x^j \partial_i - 2x_i x^j \partial_j \quad \text{(inversions)}, \qquad X_0 = x^i\partial_i \quad \text{(homothety)}, \tag{2.1}$$

where $\partial_i = \partial/\partial x^i$ with $i, j = 1, \dots, n$; $x_i = g_{ij}x^j$; and the flat metric $g = \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$ has a trace $p - q$ $(p + q = n$, sum over repeated indices is understood).

2.2. Theorem of isomorphism in the generic case

We can now state the main result of this work whose proof will be given in Section 5.

Theorem 2.1. *If* $n = p + q \geq 2$,
1. *there exists an isomorphism of* $\mathrm{o}(p+1, q+1)$*-modules:*

$$\sigma_{\lambda,\mu} : \mathcal{D}^3_{\lambda,\mu} \xrightarrow{\cong} S^3_\delta \tag{2.2}$$

provided

$$\delta = \mu - \lambda \notin \left\{ \frac{n+2}{2n}, \frac{n+4}{2n}, \frac{2}{n}, 1, \frac{n+1}{n}, \frac{n+2}{n}, \frac{n+3}{n}, \frac{n+4}{n} \right\}, \tag{2.3}$$

2. *for every* λ *and* μ *as in* (2.3), *this isomorphism is unique under the condition that the principal symbol be preserved at each order*.

Therefore, in the general situation, the unique invariant of the $\mathrm{o}(p+1, q+1)$-module $\mathcal{D}^3_{\lambda,\mu}$ is the difference: $\delta = \mu - \lambda$.

We will call the particular values of δ given by the formula (2.3) resonances.

Remark 2.2. Theorem 2.1 is compatible with the result of [7], but the demonstration will be different.

Proposition 2.3. *If* $n = 1$, *then Theorem* 2.1 *holds with the resonances:* $1, \frac{3}{2}, 2, \frac{5}{2}, 3$. (*see* [9,11]).

Definition 2.4. The isomorphism $\sigma_{\lambda,\mu}$ given by (2.2) is called the conformally equivariant symbol map while its inverse

$$Q_{\lambda,\mu} : S^3_\delta \to \mathcal{D}^3_{\lambda,\mu}, \qquad (2.4)$$

will be called the conformally equivariant quantization map.

2.3. *The modules $\mathcal{D}^3_{\lambda,\mu}$ in the resonant case*

We study in this section the singular modules corresponding to the resonances in the case $n \geq 2$ (for the $n = 1$ case, see [9]).

Theorem 2.5. *For each resonant value of δ, there exists a unique pair (λ, μ) of weights such that the $\mathrm{o}(p+1, q+1)$-modules $\mathcal{D}^3_{\lambda,\mu}$ and S^3_δ are isomorphic, as given below:*

δ	λ	μ
1	0	1
$2/n$	$(n-2)/2n$	$(n+2)/2n$
$(n+1)/n$	0	$(n+1)/n$
	$-(1/n)$	1
$(n+2)/n$	$-(1/n)$	$(n+1)/n$
$(n+3)/n$	$-(2/n)$	$(n+1)/n$
	$-(1/n)$	$(n+2)/n$
$(n+4)/n$	$-(2/n)$	$(n+2)/n$
$(n+2)/2n$	0	$(n+2)/2n$
	$(n-2)/2n$	1
$(n+4)/2n$	$-(1/n)$	$(n+2)/2n$
	$(n-2)/2n$	$(n+1)/n$

We will show that the isomorphism is not unique; there exists, actually, a one-, two-, or three-parameter family of such isomorphisms in each resonant case.

3. Differential operators and symbols

Consider the determinant bundle $\Lambda^n T^*M \to M$. Let us recall that a tensor density of degree λ on M is a smooth section, ϕ, of the line bundle $|\Lambda^n T^*M|^{\otimes\lambda}$. The space $\mathcal{F}_\lambda(M)$ of tensor densities of degree λ has a natural structure of a $\mathrm{Vect}(M)$-module, defined by the Lie derivative.

In coordinates:

$$\phi = \phi(x^1, \dots, x^n)|\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|^\lambda.$$

The action of $X \in \mathrm{Vect}\,\mathbb{R}^n$ on $\phi \in C^\infty(\mathbb{R}^n)$ is given by

$$L^\lambda_X \phi = X^i \partial_i \phi + \lambda \partial_i X^i \phi. \qquad (3.1)$$

Note, that this formula does not depend on the choice of local coordinates.

Remark 3.1. The simplest examples of modules of tensor densities are $\mathcal{F}_0 = C^\infty(\mathbb{R}^n)$ and $\mathcal{F}_1 = \Omega^n(\mathbb{R}^n)$, the module $\mathcal{F}_{1/2}$ is particularly important for geometric quantization (see [2]).

Consider the space $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^n)$(or $\mathcal{D}_{\lambda,\mu}$ for short) of differential operators on tensor densities, $A : \mathcal{F}_\lambda \to \mathcal{F}_\mu$. The natural $\mathrm{Vect}(\mathbb{R}^n)$-action on $\mathcal{D}_{\lambda,\mu}$ is given by

$$L_X^{\lambda,\mu}(A) = L_X^\mu \circ A - A \circ L_X^\lambda. \tag{3.2}$$

Denote $\mathcal{D}^k_{\lambda,\mu} \subset \mathcal{D}_{\lambda,\mu}$ the $\mathrm{Vect}(\mathbb{R}^n)$ — module of kth-order differential operators. In local coordinates any linear differential operator of order k is of the form:

$$A = a_k^{i_1\ldots i_k}\partial_{i_1}\partial_{i_2}\ldots\partial_{i_k} + \cdots + a_1^i\partial_i + a_0,$$

with coefficients $a_k^{i_1\ldots i_k} \in C^\infty(\mathbb{R}^n)$.

An operator $A : \mathcal{F}_\lambda \to \mathcal{F}_\mu$ is called a local operator on M if for all $\phi \in \mathcal{F}_\lambda$ one has $\mathrm{Supp}(A(\phi)) \subset \mathrm{Supp}(\phi)$. It is a classical result (see [15,16]) that such operators are in fact locally given by differential operators.

Consider the space $\mathrm{Pol}(T^*\mathbb{R}^n)$ of functions on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$, polynomial on the fibers: $P(\xi) = \sum_{l=0}^k \bar{a}_l^{i_1\ldots i_l}\xi_{i_1}\ldots\xi_{i_l}$. One defines a one parameter family of $\mathrm{Vect}(\mathbb{R}^n)$-modules on the space of symbols by $S_\delta := \mathrm{Pol}(T^*M) \otimes \mathcal{F}_\delta$.

The action of $\mathrm{Vect}(\mathbb{R}^n)$ on S_δ reads:

$$L_X^\delta(P) = \left(X^i\partial_i - \xi_j\partial_i(X^j)\frac{\partial}{\partial_{\xi_i}}\right)(P) + \delta(\partial_i X^i)P. \tag{3.3}$$

Throughout this paper, we will identify the space of symbols with the space of symmetric contravariant tensor fields on $\mathbb{R}^n$.

3.1. *Identification of the vector spaces* $\mathcal{D}(\mathbb{R}^n)$ *and* $\mathrm{Pol}(T^*\mathbb{R}^n)$

Let us consider the following map:

$$\begin{aligned}\sigma : \mathcal{D}(\mathbb{R}^n) \to \mathrm{Pol}(T^*(\mathbb{R}^n)),\ & a_k^{i_1,\ldots,i_k}\partial_{i_1}\partial_{i_2}\ldots\partial_{i_k} + \cdots + a_1^i\partial_i + a_0 \\ & \mapsto a_k^{i_1,\ldots,i_k}\xi_{i_1}\ldots\xi_{i_k} + \cdots + a_0.\end{aligned} \tag{3.3$'$}$$

We obtain in this way an isomorphism of vector spaces $\mathcal{D}(\mathbb{R}^n) \cong \mathrm{Pol}(T^*\mathbb{R}^n)$.

The $\mathrm{Vect}(\mathbb{R}^n)$-actions are, of course, different. We will, therefore, distinguish two Vect $(\mathbb{R}^n)$-modules

$$\mathcal{D}_{\lambda,\mu} \equiv (\mathrm{Pol}(T^*(\mathbb{R}^n)), L_X^{\lambda,\mu}), \tag{3.4}$$

$$S_\delta \equiv (\mathrm{Pol}(T^*(\mathbb{R}^n)), L_X^\delta), \delta = \mu - \lambda. \tag{3.5}$$

Remark 3.2. In mathematical physics this identification is often called normal ordering. Another frequently used way to identify the space of differential operators on $\mathbb{R}^n$ with the space $\mathrm{Pol}(T^*\mathbb{R}^n)$ is provided by the Weyl symbol calculus.

3.2. Comparison of the Vect$(\mathbb{R}^n)$*-action on* $\mathcal{D}_{\lambda,\mu}$ *and* S_δ

Let us compare the Vect$(\mathbb{R}^n)$-action on $\mathcal{D}_{\lambda,\mu}$ with the standard Vect$(\mathbb{R}^n)$-action on Pol$(T^*\mathbb{R}^n)$. We will use the preceding identification and write the Vect$(\mathbb{R}^n)$-action (3.2) in term of polynomials.

Lemma 3.3. *The* Vect$(\mathbb{R}^n)$*-action on* $\mathcal{D}_{\lambda,\mu}$ *has the following form:*

$$L_X^{\lambda,\mu} = L_X^\delta - \tfrac{1}{2}\partial_{ij}X\partial_{\xi_i}\partial_{\xi_j} - \lambda(\partial_i \circ D)X\partial_{\xi_i}, \tag{3.6}$$

where $DX = \partial_i X^i$.

Proof. Straightforward computation. □

Proposition 3.4. *The action of* $\mathrm{o}(p+1, q+1)$ *on* $\mathcal{D}_{\lambda,\mu}$ *reads:*

1. $L_X^{\lambda,\mu} = L_X^\delta$ *for all* $X \in ce(p,q) = \langle X_0, X_{ij}, X_i\rangle$, (3.7)
2. $L_{X_i}^{\lambda,\mu} = L_{X_i}^\delta - \xi_i T + 2(\mathcal{E} + n\lambda)\partial_{\xi_i}$, (3.8)
   where $T = \partial_{\xi^j}\partial_{\xi_j}$ *is the trace* $\mathcal{E} = \xi_j\partial_{\xi_j}$, *the Euler operator*.

Proof. This is a direct consequence of the preceding lemma. □

4. Explicit formulae for the isomorphism

Let us give the explicit formula for the isomorphism (2.2). The isomorphism (2.2) is uniquely determined by two properties:

1. $\sigma_{\lambda,\mu}$ intertwines the $\mathrm{o}(p+1, q+1)$-action (3.3) and (3.4):

$$\sigma_{\lambda,\mu} \circ L_X^{\lambda,\mu} = L_X^\delta \circ \sigma_{\lambda,\mu} \quad \text{for all } X \in \mathrm{o}(p+1, q+1). \tag{4.1}$$

2. $\sigma_{\lambda,\mu}$ preserve the principal symbol of A.

4.1. The image of an operator and the image of a symbol

If $A = a_3^{ijk}\partial_i\partial_j\partial_k + a_2^{ij}\partial_i\partial_j + a_1^i\partial_i + a_0 \in \mathcal{D}^3_{\lambda,\mu}$, then $\sigma_{\lambda,\mu}(A) = \bar{a}_3^{ijk}\xi_i\xi_j\xi_k + \bar{a}_2^{ij}\xi_i\xi_j + \bar{a}_1^i\xi_i + \bar{a}_0$ is given by the following proposition.

Proposition 4.1. *The isomorphism* $\sigma_{\lambda,\mu}$ *is given by*

$$\begin{aligned}
&\bar{a}_3^{ijk} = a_3^{ijk}, \quad \bar{a}_2^{ij} = a_2^{ij} + \alpha_1 g_{lm}(g^{jk}\partial_k a_3^{ilm} + g^{ik}\partial_k a_3^{jlm}) + \alpha_2\partial_k a_3^{ijk} + \alpha_3 g^{ij}g_{lm}\partial_k a_3^{klm},\\
&\bar{a}_1^i = a_1^i + \alpha_4 g^{ij}g_{kl}\partial_j a_2^{kl} + \alpha_5\partial_j a_2^{ij} + \alpha_6 g^{jk}g_{lm}\partial_j\partial_k a_3^{ilm} + \alpha_7\partial_j\partial_k a_3^{ijk}\\
&\qquad + \alpha_8 g^{ij}g_{lm}\partial_j\partial_k a_3^{klm},\\
&\bar{a}_0 = a_0 + \alpha_9\partial_i a_1^i + \alpha_{10}g^{ij}g_{kl}\partial_i\partial_j a_2^{kl}\alpha_{11}\partial_i\partial_j a_2^{ij} + \alpha_{12}g^{ij}g_{lm}\partial_i\partial_j\partial_k a_3^{klm}\\
&\qquad + \alpha_{13}\partial_i\partial_j\partial_k a_3^{ijk},
\end{aligned} \tag{4.2}$$

where

$$\alpha_1 = \frac{-3n(-1+\delta+2\lambda)}{2(n\delta-2)(n\delta-4-n)}, \qquad \alpha_2 = \frac{3(n\lambda+2)}{n\delta-n-4},$$
$$\alpha_3 = \frac{-3n(2\lambda+\delta-1)}{(n\delta-2)(n\delta-n-2)(n\delta-n-4)}, \qquad \alpha_4 = \frac{-n(2\lambda+\delta-1)}{(n\delta-2)(n\delta-n-2)},$$
$$\alpha_5 = \frac{2(n\lambda+1)}{n\delta-n-2}, \qquad \alpha_6 = \frac{-3(n\lambda+1)(n\lambda-n+n\delta-1)}{(2n\delta-n-4)(n\delta-n-3)(n\delta-n-2)},$$
$$\alpha_7 = \frac{3(n\lambda+2)(n\lambda+1)}{(n\delta-n-2)(n\delta-n-3)},$$
$$\alpha_8 = \frac{-3(n\lambda+1)(4n^2\lambda\delta-2n^2\lambda-3n^2\delta+n^2+2n^2\delta^2-10n\lambda+5n-4n\delta-2)}{(n\delta-2)(n\delta-n-2)(n\delta-n-3)(2n\delta-n-4)},$$
$$\alpha_9 = \frac{\lambda}{\delta-1}, \qquad \alpha_{10} = \frac{-n\lambda(\delta+\lambda-1)}{(\delta-1)(n\delta-n-1)(2n\delta-n-2)},$$
$$\alpha_{11} = \frac{\lambda(n\lambda+1)}{(\delta-1)(n\delta-1-n)}, \qquad \alpha_{12} = \frac{-3n\lambda(n\lambda+1)(\delta+\lambda-1)}{(\delta-1)(n\delta-1-n)(n\delta-n-2)(2n\delta-2-n)},$$
$$\alpha_{13} = \frac{\lambda(n\lambda+1)(n\lambda+2)}{(\delta-1)(n\delta-n-1)(n\delta-2-n)}. \tag{4.3}$$

Proposition 4.2. *The inverse* $\mathrm{o}(p+1,q+1)$*-equivariant quantization map* $Q_{\lambda,\mu}$ *is given by*

$$\begin{aligned}
a_3^{ijk} &= \bar{a}_3^{ijk}, \quad a_2^{ij} = \bar{a}_2^{ij} - \alpha_1 g_{lm}(g^{jk}\partial_k\bar{a}_3^{ilm} + g^{ik}\partial_k\bar{a}_3^{jlm}) - \alpha_2\partial_k\bar{a}_3^{ijk} - \alpha_3 g^{ij}\partial_k\bar{a}_3^{klm}g_{lm},\\
a_1^i &= \bar{a}_1^i - \alpha_4 g^{ij}\partial_j\bar{a}_2^{kl}g_{kl} - \alpha_5\partial_j\bar{a}_2^{ij} + (\alpha_1\alpha_5-\alpha_6)g^{jk}\partial_j\partial_k\bar{a}_3^{ilm}g_{lm}\\
&\quad +(\alpha_2\alpha_5-\alpha_7)\partial_j\partial_k\bar{a}_3^{ijk} + ((2\alpha_1+\alpha_2+n\alpha_3)\alpha_4 + (\alpha_1+\alpha_3)\alpha_5 - \alpha_8)g^{ij}\partial_j\partial_k\bar{a}_3^{klm}g_{lm},\\
a_0 &= \bar{a}_0 - \alpha_9\partial_i\bar{a}_1^i + (\alpha_4\alpha_9-\alpha_{10})g^{ij}\partial_i\partial_j\bar{a}_2^{kl}g_{kl} + (\alpha_5\alpha_9-\alpha_{11})\partial_i\partial_j\bar{a}_2^{ij}\\
&\quad +((2\alpha_1+\alpha_2+n\alpha_3)(\alpha_{10}-\alpha_4\alpha_9) + (2\alpha_1+\alpha_3)(\alpha_{11}-\alpha_5\alpha_9)\\
&\quad +(\alpha_6+\alpha_8)\alpha_9-\alpha_{12})g^{ij}\partial_i\partial_j\partial_k\bar{a}_3^{klm}g_{lm}\\
&\quad +((\alpha_7-\alpha_2\alpha_5)\alpha_9+\alpha_2\alpha_{11}-\alpha_{13})\partial_i\partial_j\partial_k\bar{a}_3^{ijk}.
\end{aligned} \tag{4.4}$$

4.2. *Particular case corresponding to* $\mathcal{D}^2_{\lambda,\mu}$ *and* S^2_δ

If $\bar{a}_3^{ijk} = 0$, then we obtain

$$\bar{a}_2^{ij} = a_2^{ij}, \qquad \bar{a}_1^i = a_1^i + \alpha_4 g^{ij}g_{kl}\partial_j a_2^{kl} + \alpha_5\partial_j a^{ij},$$
$$\bar{a}_0 = a_0 + \alpha_9\partial_i a_1^i + \alpha_{10}g^{ij}g_{kl}\partial_i\partial_j a_2^{kl} + \alpha_{11}\partial_i\partial_j a_2^{ij}$$

and

$$a_2^{ij} = \bar{a}_2^{ij}, \qquad a_1^i = \bar{a}_1^i - \alpha_4 g^{ij}\partial_j\bar{a}_2^{kl}g_{kl} - \alpha_5\partial_j\bar{a}_2^{ij},$$
$$a_0 = \bar{a}_0 - \alpha_9\partial_i\bar{a}_1^i + (\alpha_4\alpha_9-\alpha_{10})g^{ij}\partial_i\partial_j\bar{a}_2^{kl}g_{kl} + (\alpha_5\alpha_9-\alpha_{11})\partial_i\partial_j\bar{a}_2^{ij}.$$

We recognize the result of [6].

4.3. Application

Let us illustrate our quantization procedure with a specific and important example, namely the quantization of the geodesic flow on a conformally flat manifold $(M, \bar{g})$. Consider, on T^*M, the quadratic Hamiltonian

$$H = \bar{g}^{ij}\xi_i\xi_j,$$

whose flow projects onto the geodesics of $(M, \bar{g})$.

In the case $n \geq 2$, and for $\lambda = \mu = \frac{1}{2}$, the quantization map $Q_{\lambda,\mu}$ yields the following expression:

$$Q_{1/2,1/2}(H) = \triangle - \frac{n^2}{4(n-1)(n+2)} R,$$

where $\triangle$ is the Laplace operator and R the scalar curvature of $(M, \bar{g})$. This operator is a natural new candidate for the quantized Hamiltonian of the geodesic flow on a (pseudo-) Riemannian manifold. None of the expressions obtained in the literature by different methods of quantization (see, e.g., [5] for the relevant references) correspond to this one; all these expressions lack the conformal equivariance property (in the conformally flat case).

5. Conformally equivariant quantization map

It should be emphasized that the isomorphism (2.2) is necessarily given by a differential map. This fact is already guaranteed by the equivariance with respect to the subalgebra $\mathbb{R} \ltimes \mathbb{R}^n$ generated by homotheties and translations, i.e. by the following proposition.

Proposition 5.1 (Lecomte and Ovsienko [13]). *If $k \geq l$, any $\mathbb{R} \ltimes \mathbb{R}^n$-equivariant map $S^k_\delta \to S^l_\delta$ is local.*

By Peetre's theorem [15,16] such maps are locally given by differential operators. We will solve the equivariance equation and show that the quantization map is given by a globally defined differential operator.

Proposition 5.1 together with the generalized Weyl–Brauer theorem (cf. [6]), leads to the general form for a $e(p,q)$-equivariant quantization map $Q_{\lambda,\mu} : S^k_\delta \to \mathcal{D}^k_{\lambda,\mu}$ given by differential operators $Q_{\lambda,\mu} = \alpha_{r,e,g,d,l,t} R^r E^e G^g D^d \triangle^l \operatorname{Tr}^t$, where $\alpha_{r,e,g,d,l,t}$ are smooth function on M and

$$R = \xi^i\xi_i, \qquad E = \mathcal{E} + \frac{n}{2} = \xi_i \frac{\partial}{\partial_{\xi_i}} + \frac{n}{2}, \qquad G = \xi^i \frac{\partial}{\partial x^i},$$
$$D = \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial x^i}, \qquad \triangle = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x_i}, \qquad T = \frac{\partial}{\partial \xi^i}\frac{\partial}{\partial \xi_i},$$

generate the commutant $(e(p,q) = \langle X_{ij}, X_i\rangle)^!$ in $\operatorname{End}(\mathbf{C}[\xi_1, \dots, \xi_n, x^1, \dots, x^n])$. Since $[Q_{\lambda,\mu}, L_{X_i^\delta}] = (-\partial_i(\alpha_{r,e,g,d,l}))Q_{\lambda,\mu}$ and $[Q_{\lambda,\mu}, L_{X_0^\delta}] = 2(r+g+l-t)Q_{\lambda,\mu}$, then the equivariance with respect to translations and homotheties implies that $\alpha_{r,e,g,d,l}$ are constant and $t = r + g + l$. Finally, we obtain the following proposition.

Proposition 5.2. *Any* $\mathrm{o}(p+1, q+1)$*-equivariant quantization map* $Q_{\lambda,\mu} : S^k_\delta \to \mathcal{D}^k_{\lambda,\mu}$ *is of the form*

$$Q_{\lambda,\mu} = \alpha_{r,e,g,d,l} R_0^r \mathcal{E}^e G_0^g D^d \triangle_0^l, \tag{5.1}$$

where we have put

$$R_0 = R \circ T, \qquad G_0 = G \circ T, \qquad \triangle_0 = \triangle \circ T.$$

We will also impose the natural normalization condition which demands that the principal symbol be preserved:

$$\alpha_{r,e,0,0,0} = \begin{cases} 1 & \text{if } (r,e) = (0,0), \\ 0 & \text{otherwise}. \end{cases} \tag{5.1$'$}$$

5.1. Solving the equivariance equation

In the case of third order differential operators, which is the one this article is devoted to, the corresponding form is given by the following proposition:

Proposition 5.3. *There exists a unique quantization map of the form:*

$$\begin{aligned} Q_{\lambda,\mu} = {} & \mathrm{Id} + \gamma_1 G_0 + \gamma_2 D + \gamma_3 \mathcal{E} D + \gamma_4 \triangle_0 + \gamma_5 D^2 + \gamma_6 \mathcal{E} D^2 + \gamma_7 {G_0}^2 + \gamma_8 \mathcal{E} \triangle_0 \\ & + \gamma_9 D \triangle_0 + \gamma_{10} D^3 + \gamma_{11} \mathcal{E} G_0 + \gamma_{12} \mathcal{E}^2 D + \gamma_{13} R_0 D, \end{aligned} \tag{5.2}$$

satisfying the equivariance equation

$$Q_{\lambda,\mu} L^\delta_X = L^{\lambda,\mu}_X Q_{\lambda,\mu} \quad \text{for all } X \in \mathrm{o}(p+1, q+1). \tag{5.3}$$

The equivariance condition for the quantization map leads to the following system:

$$\begin{aligned} & (2 - n\delta)(\gamma_1 + \gamma_{11}) - (\gamma_2 + \gamma_3 + \gamma_{12}) = -\tfrac{1}{2}, \\ & (2 + n(1 - 2\delta))\gamma_4 - \gamma_5 = n\lambda(\gamma_1 + \gamma_{11}), \\ & 2(1 + n(1-\delta))\gamma_5 = n\lambda(\gamma_2 + \gamma_3 + \gamma_{12}), \qquad (1-\delta)\gamma_2 = \lambda, \\ & 2\gamma_3 + (6 + n(1-\delta))\gamma_{12} = 0, \qquad 2\gamma_2 + n(1-\delta)\gamma_3 - 4\gamma_{12} = 1, \\ & -\gamma_3 + (2 - n\delta)\gamma_{11} - 3\gamma_{12} = 0, \qquad \gamma_1 + 2\gamma_{11} + (2 + n(1-\delta))\gamma_{13} = 0, \\ & 2(2\gamma_5 + (3 + n(1-\delta))\gamma_6) = \gamma_2 + (n\lambda + 2)\gamma_3 + (3n\lambda + 4)\gamma_{12}, \\ & 2\gamma_4 - \gamma_6 + (4 + n(1-2\delta))\gamma_8 = \gamma_1 + (n\lambda + 2)\gamma_{11}, \\ & 2\gamma_4 + 2(2 + n(1-\delta))\gamma_7 + 2\gamma_8 = (n\lambda + 1)(\gamma_1 + 2\gamma_{11}), \\ & (2 + n(1-\delta))\gamma_9 = n\lambda(\gamma_4 + \gamma_8), \qquad 3(2 + n(1-\delta))\gamma_{10} = n\lambda(\gamma_5 + \gamma_6). \end{aligned}$$

The solution of this system [1] is unique for every δ as in (2.3).

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[1] The system has been interpreted with Maple logiciel.

5.2. Example: $\lambda = \mu = \frac{1}{2}$

In this special case (considered in the framework of geometric quantization) one obtains:

$$Q_{1/2,1/2} = \mathrm{Id} + \frac{1}{2}D + \frac{n}{8(n+1)(n+2)}\,\triangle_0 + \frac{n}{8(n+1)}D^2 + \frac{1}{4(n+1)(n+3)}\mathcal{E}D^2 - \frac{n^2+2n-2}{4(n+1)(n+2)(n+3)(n+4)}\mathcal{E}\,\triangle_0 + \frac{n}{16(n+3)(n+4)}D\,\triangle_0 - \frac{1}{8(n+3)(n+4)}{G_0}^2 + \frac{n}{48(n+3)}D^3.$$

Remark 5.4. If $n \to \infty$, the preceding formula becomes:

$$Q_{1/2,1/2} = \mathrm{Id} + \tfrac{1}{2}D + \tfrac{1}{8}D^2 + \tfrac{1}{48}D^3.$$

We recognize the Weyl quantization (cf. [8], p. 87; [1]):

$$Q_{\mathrm{Weyl}} = \exp\left(i(\tfrac{1}{2}\hbar)D\right) = \mathrm{Id} + i\tfrac{1}{2}\hbar D - (\tfrac{1}{8}\hbar^2)D^2 - i(\tfrac{1}{48}\hbar^3)D^3 + \mathrm{O}(\hbar^4).$$

5.3. Proof of Theorem 2.1

The $\mathrm{o}(p+1, q+1)$-equivariant quantization map (5.2) coincides with the expression (4.4) according to the identification given by σ (see $(3.3)'$). We have thus proven the existence of an isomorphism (2.2) provided the coefficients α are well defined, i.e. condition (2.3) holds. This proves part (1) of Theorem 2.1.

Then, the formula (5.1) and the normalization condition insure that, up to a multiplicative constant, every $\mathrm{o}(p+1, q+1)$-equivariant quantization map is, indeed, of the form (5.2). The uniqueness of the quantization map (part (2) of Theorem 2.1) follow immediately from (5.2).

5.4. Proof of Theorem 2.5

The previous system determines all $\mathrm{o}(p+1, q+1)$-equivariant linear maps form S^3_δ to $\mathcal{D}^3_{\lambda,\mu}$. In the resonant case, of course, this system has, in general, no solution. However, solving it for γ and λ as an extra indeterminate, one immediately obtains the values of λ and μ as shown in the table.

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